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by

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On Satellite Umbra/Penumbra Entry and Exit Positions

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Abstract

The problem of computing Earth satellite entry and exit positions through the Earth's umbra and penumbra, for satellites in elliptical orbits, is solved without the use of a quartic equation. **A** condition for existence of a solution is given. This problem is related to perturbation. force resulting from solar radiation pressure.

Keywords: umbra/penumbra, Halley's method, Newton's method

1 Introduction

The problem of computing Earth satellite (in elliptical orbits) entry and exit positions through the Earth's umbra and penumbra is a problem dating from the earliest days of the space age, but it is still of the utmost importance to many space projects for thermal and power considerations (Mullins, 1991). It's also important for optical tracking of a satellite. To a lesser extent, the satellite external torque history and the sensor systems are influenced by the time the satellite spends in the Earth's shadow.

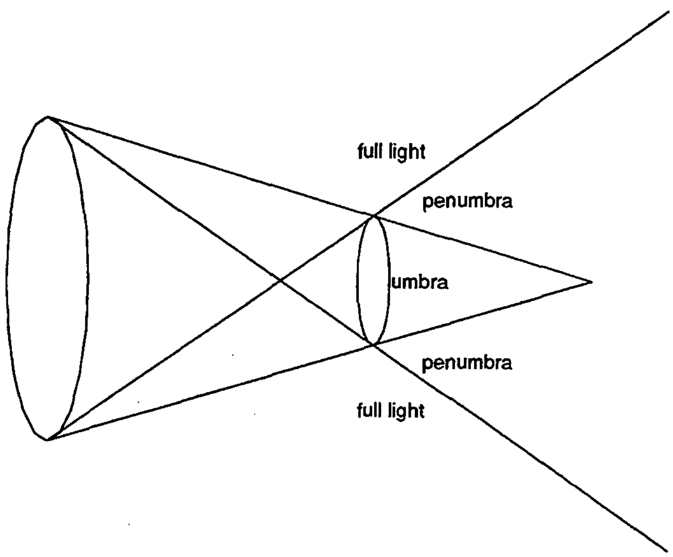

Figure 1: Earth umbra and penumbra

The umbra is the conical total shadow projected from the Earth on the side opposite the Sun. In this region, the intensity of the solar radiation is zero. The penumbra is the partial shadow between the umbra and the full-light region (see figure 1). In the penumbra, the light of the Sun is only partially cut off by the Earth, and the intensity is between 0 and 1. All textbooks discussing the problem (e.g. Geyling and Westerman, 1971, and Escobal, 1985) even the recent work by Mullins (1991), suggest the use of a quartic equation analytic solution. Because the quartic is a result of squaring the equation of interest, one must check all four solutions and discard the spurious ones. In this paper, we examine solving the original equation numerically. We will give **a** condition for the existence of a solution, discuss the initial guess for the iterative scheme, and compare the complexity of the two schemes (ours versus the analytic solution of the quartic).

The shadow problem has been solved in the past by assuming a cylindrical shadow behind the Earth (Geyling and Westerman, 1971), or a conical shadow which is more realistic (Fixler, 1964, and Mullins, 1991). The numerical solution will be discussed for each case.

2 Problem Formulation

In this section, we formulate the problem using both cylindrical and conical shadow geometry. We'll see that the solution method is different in the two cases.

2.1 Cylindrical Shadow

In this case the orbital geometry is given in figure 2 (Escobal, 1985, p. 157, or Vallado, 1996, p.521).

The analysis given by Escobal (1985) and Vallado (1996) show that the true anomaly, ν, at entrance and exit into the shadow satisfies the following equation:

$$R_{\oplus}^2(1 + e\cos\nu)^2 + p^2(\beta_1\cos\nu + \beta_2\sin\nu)^2 - p^2 = 0 \tag{1}$$

where $R_{\oplus}$ is the radius of the Earth ($\sim$ 6378.136 km), $R_{\odot}$ is the Sun's position vector ($\sim$ 696000 km), p is semi-parameter, and e is the eccentricity.

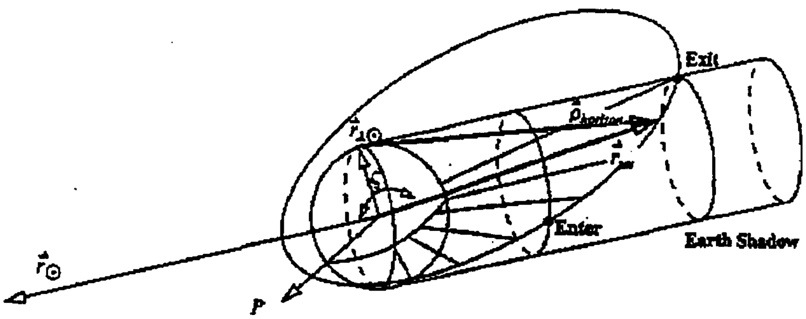

Figure **2:** Cylindrical Shadow

The remaining classical orbital elements are inclination, i, longitude of the ascending node, Ω, and the argument of perigee, w. The parameters β_1 and β_2 are given by

$$\beta_1 = \frac{\vec{R_\odot} \cdot \vec{P}}{R_\odot}$$

$$\beta_2 = \frac{\vec{R_\odot} \cdot \vec{Q}}{R_\odot}$$

The unit vectors $\vec{P}$ and $\vec{Q}$ are defined by

$$\vec{P} = \left| \begin{array}{l} \cos w \cos\Omega - \sin w \sin\Omega \cos i \\ \cos w \sin\Omega + \sin w \cos\Omega \cos i \\ \sin w \sin i \end{array} \right]$$

$$\vec{Q} = \left| \begin{array}{l} -\sin\omega\cos\Omega - \mathrm{coswsinflcosi} \\ -\sin\omega\sin\Omega + \cos\omega\cos\Omega\cos i \\ \cos w \sin i \end{array} \right]$$

For circular orbits and if $i = 0, \pi$, $\vec{P}$ should be redefined in a convenient manner (see Escobal, 1985).

2.2 Conical Shadow

In this case, one must distinguish between umbra (full shadow) and penumbra (partial shadow) regions. In the umbra case, we must solve a system of two nonlinear equations (see Mullins, 1991). The first equation models the surface of the shadow cone

$$F(x_{\rm sh}, y_{\rm sh}, z_{\rm sh}) = y_{\rm sh}^2 + z_{\rm sh}^2 - (d - x_{\rm sh})^2 \tan^2 \sigma = 0 \tag{2}$$

where d is the distance from center of the Earth to apex of shadow cone ($\sim$ **1.3836**$\cdot 10^6$ km), and σ is half angle of that cone ($\sim .26412°$). The second equation describes the orbit

$$G(x_0, y_0) = \left(\frac{x_0 + ae}{a}\right)^2 + \left(\frac{y_0}{b}\right)^2 - 1 = 0 \tag{3}$$

where $b = a\sqrt{1-e^2}$. Because the two equations are *not* in the same coordinate system, we take $\vec{r}_{\rm sh}$ and rotate it to get $\vec{r}_0$. The transformation is given by

$$\vec{r}_0 \;=\; ROT3(\omega)\, ROT1(i)\, ROT3(\Omega)\, ROT1(-\epsilon)\, ROT3(\pi - L)\, \vec{r}_{\rm sh}$$

where ϵ is the mean obliquity of the ecliptic ($\sim$ **23.5"**), L is the ecliptic longitude of the Sun, and $ROT1(\phi), ROT3(\phi)$ are rotations about the x, z axis (respectively) by ϕ. If we denote the transformation matrix by $\boldsymbol{A}$, then

$$x_{\rm sh} = a_{11}x_0 + a_{21}y_0 \tag{4}$$

$$y_{\rm sh} = a_{12}x_0 + a_{22}y_0 \tag{5}$$

$$z_{\rm sh} = a_{13}x_0 + a_{23}y_0. \tag{6}$$

Notice that z_0 is zero at the intersection of the two equations **(2)-(3).** Because only solutions with $x_{\rm sh} > 0$ are acceptable(see figure 2), we must satisfy

$$a_{11}x_0 + a_{21}y_0 > 0. \tag{7}$$

Substituting **(4)-(6)** into (2), we get the following equation

$$F_1(x_0, y_0) = \alpha_0 x_0^2 + \alpha_1 y_0^2 + 2\alpha_2 x_0 y_0 + \alpha_3 x_0 + \alpha_4 y_0 - d^2 \tan^2 \sigma = 0. \tag{8}$$

where

$$\alpha_0 = a_{12}^2 + a_{13}^2 - \mathbf{a}_{,,}^2 \tan^2 \sigma$$
$$\alpha_1 = a_{22}^2 + a_{23}^2 - a_{21}^2 \tan^2 \sigma$$

$$\alpha_2 = a_{12}a_{22} + a_{13}a_{23} - a_{11}a_{21}\tan^2\sigma$$
$$\alpha_3 = 2a_{11}d\tan^2\sigma$$
$$\alpha_4 = 2a_{21}d\tan^2\sigma$$

This equation should be solved with **(3)** and (7).

Mullins **(1991)** suggests solving **(8)** subject to **(3)** and (7), using a quartic equation for x and then checking each of the four solutions with solutions of a quadratic equation for y as a function of x. Mullins admits: "The coefficients (of the quartic) are messy functions of the angles shown ...". In section 7, we show **a** better way to solve the problem without going through a quartic equation and thus without computing these "messy coefficients."

In the penumbra case, Mullins **(1991)** shows that **(2)** becomes

$$F(x_{\text{sh}}, y_{\text{sh}}, z_{\text{sh}}) = y_{\text{sh}}^2 + z_{\text{sh}}^2 - (d' + x_{\text{sh}})^2 \tan^2\sigma' = 0$$

where d' is the distance from the center of the Earth to the apex of the cone between the Sun and the Earth ($\sim$ **1.35849** $\cdot 10^6$ km), and σ' is half angle of that cone ($\sim$ **0.26901'**). This leads to an equation similar to (8) to solve. The idea presented in section 7 will be used here too.

3 Complexity of Quartic Solution.

The problem (for cylindrical shadow) can be solved using the quartic equation

$$A_0\cos^4\nu + A_1\cos^3\nu + A_2\cos^2\nu + A_3\cos\nu + A_4 = 0 \qquad (9)$$

analytically and then rejecting the spurious roots based on the following conditions: The physical solution, should satisfy the original equation and

$$\beta_1\cos\nu + \beta_2\sin\nu < 0.$$

The coefficients of the quartics are given by:

$$
\begin{aligned}
A_0 &= \left(\frac{R_\oplus}{p}\right)^4 e^4 - 2\left(\frac{R_\oplus}{p}\right)^2 \left(\beta_2^2 - \beta_1^2\right) e^2 + \left(\beta_1^2 + \beta_2^2\right)^2 \\
A_1 &= 4\left(\frac{R_\oplus}{p}\right)^4 e^3 - 4\left(\frac{R_\oplus}{p}\right)^2 \left(\beta_2^2 - \beta_1^2\right) e \\
A_2 &= 6\left(\frac{R_\oplus}{p}\right)^4 e^2 - 2\left(\frac{R_\oplus}{p}\right)^2 \left(\beta_2^2 - \beta_1^2\right) - 2\left(\frac{R_\oplus}{p}\right)^2 \left(1 - \beta_2^2\right)^2 e^2 + \\
& \quad 2\left(\beta_2^2 - \beta_1^2\right)\left(1 - \beta_2^2\right) - 4\beta_1^2\beta_2^2 \\
A_3 &= 4\left(\frac{R_\oplus}{p}\right)^4 e - 4\left(\frac{R_\oplus}{p}\right)^2 \left(1 - \beta_2^2\right) e \\
A_4 &= \left(\frac{R_\oplus}{p}\right)^4 - 2\left(\frac{R_\oplus}{p}\right)^2 \left(1 - \beta_2^2\right) + \left(1 - \beta_2^2\right)^2
\end{aligned}
$$

If the work is done economically, one finds that the number of multiplications and divisions required to compute the coefficients of the quartic is **38.** To find the solution of the quartic requires 64 multiplications/division, **5** square roots, **4** cube roots, 1 arccosine and **3** cosine evaluations. The cosine and arccosine evaluations are required only if the 'discriminant is negative, see Abramowitz and Stegun (1965).

4 Numerical Solution for Cylindrical Shadow

To solve the shadow equation (1) numerically, we can use either bracketing or fixed-point type methods. In the following, we describe only Newton's and Halley's methods which are of fixed-point type. It is first suggested to check the existence of a solution. First, rewrite (1) as:

$$f(x) = Ax^2 + Bx + Cx\sqrt{1-x^2} + D = 0 \tag{10}$$

where $x = \cos\gamma$. In order to have a solution, we must have

$$f(-1)f(1) \le 0. \tag{11}$$

Clearly equality means that $\cos \nu$ is f1. The strict inequality in (11) is equivalent to

$$1 - \left(\frac{R_\oplus}{a(1+e)}\right)^2 > \beta_1^2 > 1 - \left(\frac{R_\oplus}{a(1-e)}\right)^2$$

Note that there is *no* condition on β_2.

4.1 Newton's Method

To solve a nonlinear equation $f(x) = 0$ via Newton's method, we require an initial guess x_0. Then an iterative procedure can be followed to construct a sequence of estimates x_n, by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, \ldots$$

The iterative process converges if either

$$|f(x_n)| < Tol_f$$

or

$$|x_{n+1} - x_n| < Tol_x$$

for given tolerances. In either case we take x_{n+1} as the root. The convergence rate is quadratic. If the iterative process doesn't converge in a certain number of iterations, we stop. In this case we suggest bracketing methods. Newton's method will diverge if we hit a point where $f'(x)$ is very small.

4.2 Halley's Method

Halley's method converges faster (third order compare to second order for Newton). The iterative process is

$$x_{n+1} = x_n - \frac{f(x_n)}{\dfrac{f(x_n)f''(x_n)}{2f'(x_n)}}, \qquad n = 0, 1, \ldots$$

4.3 Bracketing Methods

In general, bracketing methods are slower, but they are safer, in the sense that convergence is guaranteed. For example, the bisection method starts with an initial interval containing the root, $[a_0, b_0]$. The process halves the interval at every step. After n iterations, the length of the interval containing the root is $\frac{b_0 - a_0}{2^n}$. Therefore, the number of iterations required depends on the length of the initial interval and the tolerance.

This simplistic method can be modified by using Regula Falsi (solving a linear equation at every step) or modified Regula Falsi (which is useful when the curvature of f is large enough.)

For example, we have solved (10) with $A = 1$, $B = -2$, $C = 1$, and $D = 1$. Newton's method required **5** iterations for convergence to 10^{-10}, Halley's method required **4** iterations and the bisection methods used **31** iterations. If we require a more realistic accuracy, let say 10^{-6}, then Halley's method requires **3** iterations, Newton's requires **4** iterations and the bracketing methods uses 19.

5 Initial Guess

Because the problem is to solve for $\cos\nu$, we know that the solution, if it exists, must lie in the interval $[-1, 1]$. For bracketing methods we suggest using this interval, and for Newton's and Halley's method, we take the midpoint of the interval, i.e. $x_0 = 0$.

For subsequent crossings through the shadow, we can take x_0 to be the previous solution.

6 Complexity of Numerical Solution

All iterative procedures require function evaluations, and some will require the evaluation of the first and maybe second derivative. The evaluation of the function requires **4** multiplications/divisions (using nested multiplication) and 1 square root. The evaluation of the first derivative is accomplished by 7 multiplications/divisions and 1 square root. The second derivative requires 8 multiplications/divisions and 1 square root. For one iteration of Halley's method we need 23 multiplications/divisions and 3 square roots. For one iteration of Newton's method we need 12 multiplications/divisions and 2 square roots. For the bisection method we need 5 multiplications/divisions and 1 square root. If we multiply the number of iterations by the cost per iteration we find that Newton's method is the cheapest with 48 multiplications/divisions and 8 square roots, then Halley's method with 69 multiplications/divisions and 9 square roots, then bisection with 95 multiplications/divisions and 5 square roots. In comparison, Newton's method is cheaper than solving the quartic and it doesn't require checking for spurious roots. Even Halley's method is competitive with the analytic solution of the quartic. We summarize the results in a table.

operation	mult./div.	sq. root	cubic root	trig. func.	number iter.
Newton	48	8	0	0	4
Halley	69	9	0	0	3
Bisection	95	5	0	0	19
Quartic	102	5	4	2*	0

Table 1: Operation count

7 Numerical Solution for Conical Shadow

In this section, we describe a numerical method to solve (8) and (3) subject to (7). We suggest guessing an initial approximation x_0 and use (3) to get the corresponding y_0

$$y_0 = \pm b\sqrt{1-e^2}. \tag{12}$$

Because (3) is quadratic, we offer here the correct sign to satisfy (7). Note that (7) describes a half plane whose boundary is a line in figures 3 and 4.

$$Y_0 = -\frac{a_{11}}{a_{21}} x_0. \tag{13}$$

'This doesn't include checking for spurious roots.

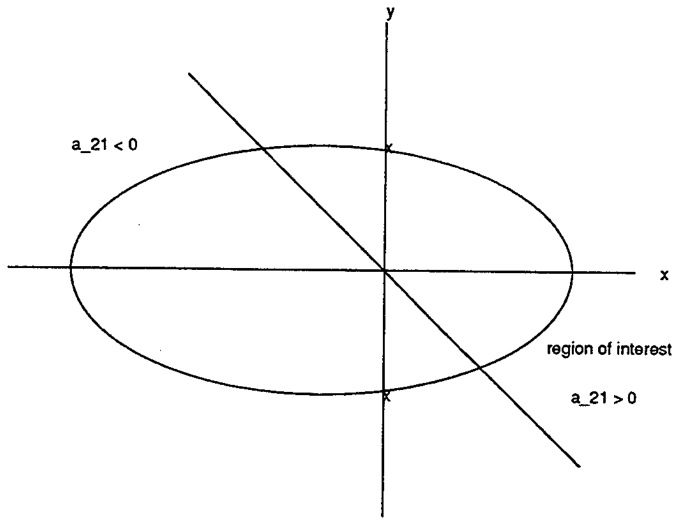

Figure **3:** a_{11} and a_{21} have the same sign

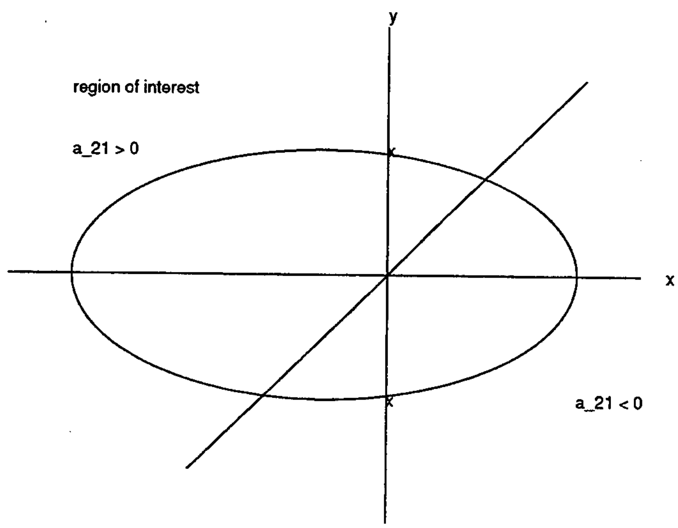

Figure **4:** a_{11} and a_{21} have opposite signs

Therefore the sign of the radical in (12) is the same as the sign of a_{21}. We now rewrite (8) **as**

$$F_1(x,y) = Ax^2 + By(x)^2 + Cxy(x) + Dx + Ey(x) + F$$

with (using **(3))**

$$y(x) = \pm\sqrt{1-e^2}\sqrt{a^2-(x+ae)^2}$$

For Newton's method, we need F_1' and y' which are given by

$$F_1'(x,y) = 2Ax + 2By(x)y'(x) + Cy(x) + Cxy'(x) + D + Ey'(x)$$

and

$$y'(x) = \mp\frac{(1-e^2)(x+ae)}{Y}$$

Now the iterative procedure is as follows

$$x_{n+1} = x_n - \frac{F_1(x_n,y_n)}{F_1'(x_n,y_n)} \qquad n = 1,2,\dots$$

$$y_{n+1} = \pm\sqrt{1-e^2}\sqrt{a^2-(x_{n+1}+ae)^2}$$

Remember to choose the sign appropriately.

8 Conclusions

In this paper, we suggest the use of iterative techniques to compute the entry and exit positions through the Earth's umbra and penumbra. We also show how to choose the initial guess for the first and subsequent crossings. Several iterative methods for the solution of the problem are compared to the currently used one. Newton's method converges fast especially at subsequent crossings, because the initial guess is close enough.

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Appendix A

This section provides two FORTRAN subroutines using Newton's method to solve both the cylndrical and cone shadows.

```
c
c       Newton's algorithm for the cylindrical shadow
c
      real*8 xn,fn,xn1,fpn,tol,tolf
      real*8 a0,a1,a2,a3
      integer indx,mxindx
c
c  max number of iterations
c
      mxindx=15
c
c    convergence tolerance on consecutive iterates
c
c     tol=1.e-18
      tol=1.e-6
c
c    convergence tolerance on closeness of function to zero
c
c     tolf=1.e-18
      tolf=1.e-6
      a0=1.
      a1=-2.
      a2=-1.
      a3=1.
c
c     initial guess
c
       xn=0.
       indx=0
```

```
5     continue
      call f(a0,a1,a2,a3,xn,fn,fpn)
c
c  is function close to zero at xn?
c
      if(dabs(fn).le.tolf) go to 20
c
c   compute next iterate
c
      xn1=xn-fn/fpn
      zn1=dsqrt(1.d0-xn1*xn1)
      indx=indx+1
      print *,indx,xn,xn1,zn1,fn
c
c   check for closeness of iterates
c
      if(dabs(xn1-xn).le.tol) go to 10
      xn=xn1
c
c   check if max number of iterates exceeded
c
      if(indx.ge.mxindx) go to 30
      go to 5
10    print *,' convergence iterates close',xn,xn1
      go to 40
20    print *,' convergence function close to zero',xn,fn
      go to 40
30    print *,' no convergence - max number of iterates'
40    stop
      end
      subroutine f(a0,a1,a2,a3,x,y,yp)
      real*8 x,y,yp
      real*8 a0,a1,a2,a3
c
c   evaluate the function
c
      z=dsqrt(1.d0-x*x)
```

```
      y=a0*x**2+a1*x+a2*x*z+a3
c
c  evaluates the first derivative
c
      yp=2.*a0*x+a1+a2*z-a2*x/z*x
      return
      end


c
c        Newton's algorithm for the cone shadow
c
      real*8 xn,yn,fn,xn1,yn1,ypn,fpn,tol,tolf
      real*8 a11,a12,a13,a21,a22,a23
      real*8 a,b,c,d,e,f
      real*8 aa,ee,e21,e21s,ae,t2s,sigma,dd,dt2s,dt2s2
      real*8 pi,ax,by,cx
      integer indx,mxindx

      pi=4.d0*datan(1.d0)
c
c  max number of iterations
c
      mxindx=35
c
c    convergence tolerance on consecutive iterates
c
c     tol=1.e-18
      tol=1.e-6
c
c    convergence tolerance on closeness of function to zero
c
c     tolf=1.e-18
      tolf=1.e-6
c
c   coefficients of transformation matrix
c
```

```
      a11=.05
      a12=.05
      a13=.05
      a21=.05
      a22=.05
      a23=.05
c  sigma half angle of shadow cone
      sigma=.26412*pi/180.
      t2s=dtan(sigma)**2
c  dd distance from center of Earth to apex of shadow cone
      dd=1.3836*10**6
      dt2s=dd*t2s
c   ee  eccentricity
      ee=.001
c   aa semi major axis
      aa=10000000
      e21=1-ee*ee
      e21s=dsqrt(e21)
      ae=aa*ee
c       print *,' aa ee ',aa,ee
c       print *,' e21 e21s ae ',e21,e21s,ae
c
c     initial guess
c
       xn=0.
       yn=aa*e21
       print *,' xn yn initially ',xn,yn

c
c     initialize counter of iteration
c
       indx=0
c
c    compute coefficients of equation to solve
c
       a=a12*a12+a13*a13-a11*a11*t2s
       b=a22*a22+a23*a23-a21*a21*t2s
```

```
      c=2.*(a12*a22+a13*a23-a11*a21*t2s)
      dt2s2=2*dt2s
      d=dt2s2*a11
      e=a21*dt2s2
      f=-d*dt2s
      print *,' a b c d e f ',a,b,c,d,e,f

5     continue
      call fcn(a,b,c,d,e,f,ae,e21,ax,by,cx,xn,yn,ypn,fn,fpn)
      print *,' indx ypn fn fpn ',indx,ypn,fn,fpn
c
c is function close to zero at xn?
c
      if(dabs(fn).le.tolf) go to 20
c
c  compute next iterate
c
      xn1=xn-fn/fpn
      yn1=e21s*dsqrt(aa*aa-(xn1+ae)**2)
      indx=indx+1
      print *,indx,xn,xn1,yn,yn1,fn
c
c  check for closeness of iterates
c
      if(dabs(xn1-xn).le.tol) go to 10
      xn=xn1
      yn=yn1
c
c  check if max number of iterates exceeded
c
      if(indx.ge.mxindx) go to 30
      go to 5
10    print *,' convergence iterates close',xn,xn1
      go to 40
20    print *,' convergence function close to zero',xn,fn
      go to 40
30    print *,' no convergence - max number of iterates'
```

```
40    stop
      end
      subroutine fcn(a,b,c,d,e,f,ae,e21,ax,by,cx,x,y,yp,fn,fpn)
      real*8 x,y,yp,fn,fpn
      real*8 a,b,c,d,e,f,ax,by,cx,ae,e21
c
c   evaluate the function
c
      ax=a*x
      by=b*y
      cx=c*x
      fn=ax*x+by*y+cx*y+d*x+e*y+f
c
c  evaluates the first derivative
c
      yp=-e21*(x+ae)/y
      fpn=2.*ax+2.*by*yp+c*y+cx*yp+d+e*yp
      return
      end
```

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